ADJUSTMENT OF GRAVIMETRIC NETWORKS*

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Abstract: We summarize the current knowledge about the adjustment of gravimetric networks in terms of the pseudo-inverse theory of least-squares processes, and give the link with the so-called free adjustment often used in practice.

1 – INTRODUCTION

Let us consider the system

$$G \underset{\sim}{p} = \underset{\sim}{d} \tag{1}$$

where:

- the vector $\underset{\sim}{d}$ represents the data point values (number m),
- the vector $\underset{\sim}{p}$ represents the parameters (number n),
- the matrix G is the design matrix linking the parameters $\underset{\sim}{p}$ to the data $\underset{\sim}{d}$.

If G is of full rank (i.e. the problem is fully constrained), then, for the overdetermined case (more observation equations than unknowns), one and only one solution exists in the least squares sense, and is given by

$$\underset{\sim}{p}^* = \left(G^T G\right)^{-1} G^T \underset{\sim}{d} \tag{2}$$

If G is not of full rank, as it is always the case for networks, then $\left(G^T G\right)^{-1}$ does not exist. This signifies that there are several solutions $\underset{\sim}{p}$, each one verifying

$$\left(G^T G\right)\underset{\sim}{p} = G^T \underset{\sim}{d} \tag{3}$$

Then we have else:
- to pick up a particular solution, from an a priori rule,
- or to modify the system (1), in order to restore unicity

$$G \underset{\sim}{p} = \underset{\sim}{d} \rightarrow G' \underset{\sim}{p}' = \underset{\sim}{d}' \tag{4}$$

i.e. with a design matrix G' of full rank.

2 – CHOICE OF A PARTICULAR SOLUTION

If several solutions $\underset{\sim}{p}$ to Eq. (3) exist, a physically sound choice is to select the one that exhibits the lowest norm, (i.e. $\left\| \underset{\sim}{p} \right\|$ minimum). This solution is given by the generalized inverse of *G*, which is the unique matrix G^+ verifying

$$G\,G^+\,G = G$$
$$G^+\,G\,G^+ = G^+$$
$$\left(G\,G^+\right)^T = G\,G^+$$
$$\left(G^+\,G\right)^T = G^+\,G$$

On practical grounds, G^+ is computed through the so-called SVD (Singular Value Decomposition) algorithm, from the expansion

$$G = U\,\Lambda\,V^T$$

where $U\,U^T = U^T\,U = I_m \qquad U \quad m \times m$

$V\,V^T = V^T\,V = I_n \qquad V \quad n \times n$

$\Lambda \quad m \times n$

and

$$\Lambda = \left[\begin{array}{cccc|c} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & 0_{q,n-q} \\ & & & \lambda_q & \\ \hline & & 0_{m-q,q} & & 0_{m-q,n-q} \end{array}\right] \qquad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q > 0 \qquad (5)$$

There are *q* non-zero values λ_i. The number *q* is the rank of matrix *G*, and the rank defect of *G* is then *n* – *q*.

We have $G^+ = V\,\Lambda^+\,U^T$, with

$$\Lambda^+ = \left[\begin{array}{cccc|c} \lambda_1^{-1} & & & & \\ & \lambda_2^{-1} & & & \\ & & \ddots & & 0_{q,m-q} \\ & & & \lambda_q^{-1} & \\ \hline & & 0_{n-q,q} & & 0_{n-q,m-q} \end{array}\right] \qquad \Lambda^+ \quad n \times m \qquad (6)$$

Furthermore, if $\underset{\sim}{d}$ is associated with the covariance matrix $C_d = \sigma_d^2\,I$, then

$$C_{\underset{\sim}{p}^*} = \left(G^+\right)\sigma_d^2\,I\left(G^+\right)^T = \sigma_d^2\left(G^T\,G\right)^+$$

and Trace $\left(\sigma_d^2\left(G^T\,G\right)^+\right)$ is minimal over the whole set of all possible solutions $\underset{\sim}{p}^*$ of Eq. (3). This is an another nice property.

In principle, the SVD algorithm permits to solve ALL least squares problems, and is part of all common least-squares software packages. But it is costly in terms of computer space and execution time, and sometimes its precision is not guaranteed for the determination of small singular values and corresponding eigenvectors. For overdetermined and underconstrained systems like the gravimetric or altimetric network problem there is an alternative technique called "free network" which consists in writing Eq. (3) as (see annex)

$$\left(G^T\ G + \Phi\right)\underset{\sim}{p} = G^T\ \underset{\sim}{d}$$

with $\Phi\ \underset{\sim}{p} = 0$, in such a way that $\left(G^T\ G + \Phi\right)^{-1}$ exists, and that $\underset{\sim}{p}^* = G^+\ \underset{\sim}{d} = \left(G^T\ G + \Phi\right)^{-1} G^T\ \underset{\sim}{d}$ (7)

The immense advantage is that the use of a standard Cholesky algorithm to compute $\underset{\sim}{p}^*$ and its associated covariance is now allowed. Of course, the disadvantage is that we have to construct Φ.

A particular value of $\underset{\sim}{p}^*$ can be also computed through iterative processes, like

$$\underset{\sim}{p}^{(n+1)} = \left(G^T\ G + \lambda^2\ I\right)^{-1}\left(G^T\ \underset{\sim}{d} + \lambda^2\ \underset{\sim}{p}^{(n)}\right),$$

with $\underset{\sim}{p}^{(1)} = \left(A^T\ A + \lambda^2\ I\right)^{-1} G^T\ \underset{\sim}{d}$, (8)

and, in fine, $\underset{\sim}{p}^* = \underset{\sim}{p}^{(\infty)}$.

But there is no longer a direct access to the covariance matrix…

If this covariance matrix is absolutely needed, G^+ can be iteratively built through

$$X^{(r+1)} = X^{(r)}\left(2I - G\ X^{(r)}\right) \quad (9)$$

with $X^{(o)} = \alpha\ G^T \qquad 0 < \alpha < 2/\lambda_1$,

where λ_1 are the largest singular values of $G\,G^T$ (the convergence can be assessed by $\left\| G\ X^{(r)} G - G \right\| \to 0$).

A last possibility to pick up a particular solution of Eq. (3) is to split the normal matrix as

$G^T\ G = M - N$ with M^{-1} well defined.

This splitting allows us to write the iteration

$$\underset{\sim}{p}^{(n)} = M^{-1} N\ p^{(n-1)} + M^{-1}\ G^T\ \underset{\sim}{d} \quad (10)$$

starting from a given $\underset{\sim}{p}^{(o)}$.

This technique can be very cheap to implement, for example by selecting $M = diag\left(G^T\ G\right)$.

The convergence of the method depends on the spectral properties of $M^{-1}\ N$ and on the particular value of the initial vector $\underset{\sim}{p}^{(o)}$.

The disadvantage is that, albeit if $\underset{\sim}{p}^{(\infty)}$ by construction strictly verifies Eq. (3), the non-unicity of the solution could be synonymous with long wavelength distortions in the network (otherwise perfectly adjusted).

3 – MODIFICATION OF THE INITIAL SYSTEM

There are two possibilities:

3.1. to complement the initial system $G \underset{\sim}{p} = \underset{\sim}{d}$ **with** $K \underset{\sim}{p} = \underset{\sim}{1}$**,**

in order to obtain the form $\begin{bmatrix} G \\ \hline K \end{bmatrix} \underset{\sim}{p} = \begin{bmatrix} \underset{\sim}{d} \\ \hline \underset{\sim}{1} \end{bmatrix} = G' \underset{\sim}{p}' = \underset{\sim}{d}'$,

where G' is now of full rank.

The normal system has then a unique solution $\underset{\sim}{p}'^*$ given by

$$\underset{\sim}{p}'^* = \left(G'^T G'\right)^{-1} G'^T \underset{\sim}{d}'$$

that can be rewritten as

$$\underset{\sim}{p}'^* = \left(G^T G + K^T K\right)^{-1} \left(G^T \underset{\sim}{d} + K^T \underset{\sim}{1}\right)$$

If furthermore we assume that $\underset{\sim}{1} = \underset{\sim}{0}$, we obtain

$$\underset{\sim}{p}'^* = \left(G^T G + K^T K\right)^{-1} G^T \underset{\sim}{d} \qquad (11)$$

Let us note the formal analogy with Eq. (7), which corresponds to $\left\| \underset{\sim}{p}'^* \right\| = \left\| \underset{\sim}{p}^* \right\|_{\min}$

The additional assumption $K = \lambda I$ with $\lambda > 0$ is of courant use, and signifies that a solution $\underset{\sim}{p}'^*$ close to zero is sought. That corresponds to seek a value of $\underset{\sim}{p}'^*$ close to zero, and that more especially as λ is large.

3.2. –to complement the normal system, the simplest form being

$$G^T G \quad \to \left[\begin{array}{c|c} G^T G & u^T \\ \hline u & \underset{\sim}{0} \end{array}\right],$$

which is equivalent to solve the system

$$\left[\begin{array}{c|c} G^T G & u^T \\ \hline u & \underset{\sim}{0} \end{array}\right] \begin{bmatrix} \underset{\sim}{p} \\ \underset{\sim}{\lambda} \end{bmatrix} = \begin{bmatrix} G^T \underset{\sim}{d} \\ \underset{\sim}{\mu} \end{bmatrix} \qquad (12)$$

or equivalently to minimize $G \underset{\sim}{p} - \underset{\sim}{d}$ under the constraint $u \underset{\sim}{p} - \underset{\sim}{\mu} = \underset{\sim}{0}$

This is the case when some data points on the network are frozen, i.e. if we suppose that the corresponding gravimetric values are perfectly known on these points.

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ANNEX – NOTION OF FREE NETWORK

From Eq. (5), one can see that

$$G^T G = V \left[\begin{array}{ccc|c} \lambda_1^2 & & & \\ & \circ & & 0_{q,n-q} \\ & & \lambda_q^2 & \\ \hline & 0_{n-q,q} & & 0_{n-q,n-q} \end{array} \right] V^T \tag{13}$$

The idea is to complete $G^T G$ in such a way that

$$\left(G^T G\right)_{completed} = V \begin{bmatrix} \lambda_1^2 & & & & & 0 \\ & \circ & & & & \\ & & \lambda_q^2 & & & \\ & & & \lambda_{q+1}^2 & & \\ & & & & \circ & \\ 0 & & & & & \lambda_n^2 \end{bmatrix} V^T \tag{14}$$

Therefore, $(G^T G)_{\text{completed}}$ exists. This is equivalent to writing

$$\left(G^T G\right)_{completed} = G^T G + \Phi$$

where $$\Phi = V \left[\begin{array}{ccc|ccc} 0 & & & & & \\ & \circ & & & 0_{q,n-q} & \\ & & 0 & & & \\ \hline & & & b_{q+1}^2 & & \\ & 0_{n-q,q} & & & \circ & \\ & & & & & \lambda_n^2 \end{array} \right] V^T \tag{15}$$

It is easy to see that $G\,\Phi = 0$ i.e. $G^T G\,\Phi = 0$. Eq. (15) suggest that Φ can be put on the form $\Phi = C^T C$, where

$$C = \overline{U} \left[\begin{array}{ccc|ccc} 0 & & & & & \\ & \bigcirc & & & 0_{q,n-q} & \\ & & 0 & & & \\ \hline & & & b_{q+1}^2 & & \\ & 0_{n-q,q} & & & \bigcirc & \\ & & & & & \lambda_n^2 \end{array} \right] V^T \qquad (16)$$

with $\overline{U}$ verifying $\overline{U}\ \overline{U}^T = \overline{U}^T\ \overline{U} = I_m$

Then

$$\underset{\sim}{p}** = \left(G^T\ G + \Phi\right)^{-1} G^T\ \underset{\sim}{d}$$

$$= V \begin{bmatrix} 1/\lambda_1^2 & & & & & \\ & \bigcirc & & & & \\ & & 1/\lambda_q^2 & & & \\ & & & 1/\lambda_{q+1}^2 & & \\ & & & & \bigcirc & \\ & & & & & 1/\lambda_n^2 \end{bmatrix} V^T\ V \left[\begin{array}{ccc|c} \lambda_1 & & & \\ & \bigcirc & & 0_{q,m-q} \\ & & \lambda_q & \\ \hline & 0_{n-q,q} & & 0_{n-q,m-q} \end{array} \right] U^T\ \underset{\sim}{d}$$

$$= V \left[\begin{array}{ccc|c} 1/\lambda_1 & & & \\ & \bigcirc & & 0_{q,m-q} \\ & & 1/\lambda_q & \\ \hline & 0_{n-q,q} & & 0_{n-q,m-q} \end{array} \right] U^T\ \underset{\sim}{d}$$

$= G^+\ \underset{\sim}{d},$ the least norm solution.

We have also $\Phi\ p** = \Phi\ p* = 0$.